Math 241

Problem Set 7 solution manual

Exercise. A7.1

a- Consider the following table:

o	1	(12)(34)	(13)(24)	(14)(23)
1	1	(12)(34)	(13)(24)	(14)(23)
(12)(34)	(12)(34)	1	(14)(23)	(13)(24)
(13)(24)	(13)(24)	(14)(23)	1	(12)(34)
(14)(23)	(14)(23)	(13)(24)	(12)(34)	1

Notice that each element is the inverse of itself (each element is of order 2). So this note, together with the above table shows that H is a subgroup.

Now to show that H is normal notice that for any $\sigma \in S_4$ we have $\sigma(ij)(kl)\sigma^{-1} = \sigma(ij)(\sigma^{-1}\sigma)(kl)\sigma^{-1} = (\sigma(ij)\sigma^{-1})(\sigma(kl)\sigma^{-1}) = (\sigma(i)\sigma(j))(\sigma(k)\sigma(l))$ which is again an element of H.

This shows that H is a normal subgroup of S_4.

b- H is isomorphic to klein4 group since it is an abelian group of order 4 , and each element is of order 2. (the table shows that H is abelian)

c- Let us first calculate the cosets of H in S_4,

we have :
$H = H_0$
$(12)H = \{(12), (34), (1324), (1423)\} = H(12) = H_1$
$(13)H = \{(13), (1234), (24), (1432)\} = H(13) = H_2$
$(23)H = \{(23), (1342), (1243), (14)\} = H_3$
$(123)H = \{(123), (134), (243), (142)\} = H_4$
$(132)H = \{(132), (234), (124), (143)\} = H_5$

and call elements of S_3 , $id = a_0$, $(12) = a_1$, $(13) = a_2$, $(23) = a_3$, $(123) = a_4$, $(132) = a_5$.

Then we define the map f from $\frac{S_4}{H} \longrightarrow S_3$ by $f(H_i) = a_i$
It is easy to see that f is a bijection by definition of f. Moreover, f homomorphism follows directly from the multiplication of cosets, since H is normal.

Hence we have an isomorphism between $\frac{S_4}{H}$ and S_3.

Section. 14

Exercise. 2

Notice that the order of $\mathbb{Z}_4 \times \mathbb{Z}_{12}$ is 48 , and the order of the subgroup $< 2 >$ in $\mathbb{Z}_4$ is 2, and the order of $< 2 >$ in $\mathbb{Z}_{12}$ is 6. Hence the order of $< 2 > \times < 2 >$ in $\mathbb{Z}_4 \times \mathbb{Z}_{12}$ is 12 .

So the order of the group is: $\frac{|\mathbb{Z}_4 \times \mathbb{Z}_{12}|}{|<2> \times <2>|} = \frac{48}{12} = 4$.

Exercise. 6

Notice that the order of $\mathbb{Z}_{12}\times\mathbb{Z}_{18}$ is 216 , and the order of the subgroup $<(4,3)>$ in $\mathbb{Z}_{12}\times\mathbb{Z}_{18}$ is 6. $(r.(4,3)=0 \Leftrightarrow 4r\equiv 0\ mod(12), 3r\equiv 0\ mod(18), \Leftrightarrow r\equiv 0\ mod(3), and\ r\equiv 0\ mod(6) \Leftrightarrow r\equiv 0\ mod(6).)$

So the order of the group is: $\frac{|\mathbb{Z}_{12}\times\mathbb{Z}_{18}|}{|<(4,3)>|}=\frac{216}{6}=36.$

Exercise. 9

First we find the subgroup $<4>=\{0,4,8\}$
Notice that $r+12\mathbb{Z}\in<4>\Leftrightarrow r\equiv 0\ mod(4)$. Then $r.(5+<4>)=0+<4>\Leftrightarrow 5r=0\ mod(4)$ $\Leftrightarrow r=0\ mod(4)$,hence the order of $5+<4>$ is 4.

Or you can use main defintion of order (the order of $5+<4>$ is the smallest positive number n such that
$n(5+<4>)=<4>$.)
$2(5+<4>)=(5+<4>)+(5+<4>)=10+<4>$,
$3(5+<4>)=(5+<4>)+(5+<4>)+(5+<4>)=15+<4>=2+<4>$,
$4(5+<4>)=8+<4>=<4>$.

Hence we deduce that the order of $(5+<4>)$ in $\frac{\mathbb{Z}_{12}}{<4>}$ is 4.

Exercise. 10

First we find the subgroup $<12>=\{0,12,24,36,48\}$
Notice that $r+60\mathbb{Z}\in<12>\Leftrightarrow r\equiv 0\ mod(12)$. Then $r.(2+<12>)=0+<12>\Leftrightarrow 2r=0\ mod(12)\Leftrightarrow r=0\ mod(6)$,hence the order of $2+<12>$ is 6.

Or using the definiton the order of $26+<12>$ is the smallest positive number n such that $n(26+<12>)=<12>$
$2(2+<12>)=(2+<12>)+(2+<12>)=4+<12>$,
$3(2+<12>)=(2+<12>)+(2+<12>)+(2+<12>)=6+<12>$,
$4(2+<12>)=8+<12>$,
$5(2+<12>)=10+<12>$,
$6(2+<12>)=12+<12>=<12>$.

Hence we deduce that the order of $(26+<12>)$ in $\frac{\mathbb{Z}_{60}}{<12>}$ is 6.

Exercise. 12

First we find the subgroup $<(1,1)>=\{(0,0),(1,1),(2,2),(3,3)\}$
Then we know that the order of $(3,1)+<(1,1)>$ is the smallest positive number n such that $n((3,1)+<(1,1)>)=<(1,1)>$
$2((3,1)+<(1,1)>)=((3,1)+<(1,1)>)+((3,1)+<(1,2)>)=(2,2)+<(1,1)>=<(1,1)>$.

Hence we deduce that the order of $(26+<12>)$ in $\frac{\mathbb{Z}_4\times\mathbb{Z}_4}{<(1,1)>}$ is 2.

Exercise. 30

m is the order of $(G:H)$ then any element $A\in\frac{G}{H}$ has an order divisible by m, and hence for any $a\in G$, $aH\in\frac{G}{H}$ then $(aH)^m=H$, but $a^mH=(aH)^m$ so $a^m\in H$.

Exercise. 31

Let $\{H_1, H_2, H_3, ...H_n\}$ be a collection of normal subgroups of G. We already know that $H = \bigcap_i H_i$ is a subgroup of G, so we only need to show it is normal.

We want to prove $gHg^{-1} \subset H$, so let $h \in H$, then $h \in H_i$ for all i, and hence since each H_i is normal subgroup then $ghg^{-1} \in H_i$for all i. This implies $ghg^{-1} \in H$, so we deduce that H is a normal subgroup.

Exercise. 32

First we let $\mathcal{X} = \{H \mid H$ is a normal subgroup of G containing $S\}$ (we know that this is a non empty set since $G \in \mathcal{X}$).

Next we consider the subgroup $N = \bigcap_{H \in \mathcal{X}} H$, then N contains S, and by the previous problem we know that N is a normal subgroup of G. Moreover, suppose K is another normal subgroup that contain S, then $K \in \mathcal{X}$, and hence by the definition of N, $N \subset K$.

So our N is the smallest normal subgroup that contain S.

Exercise. 33

Let $aC, bC \in \frac{G}{C}$, and notice that $(aC)^{-1} = a^{-1}C$.

$(aC)(bC)(aC)^{-1}(bC)^{-1}$=$(ab)C(a^{-1}b^{-1})C$=$(aba^{-1}b^{-1})C = C$ since $aba^{-1}b^{-1} \in C$. This implies that for all a, b $(aC)(bC) = (bC)(aC)$. Hence $\frac{G}{C}$ is abelian.

Section. 15

Exercise. 1

The order of $\mathbb{Z}_2 \times \mathbb{Z}_4$ is equal to 8 , and the order of (0,1) in $\mathbb{Z}_2 \times \mathbb{Z}_4$ is 4, hence the order of $\frac{\mathbb{Z}_2 \times \mathbb{Z}_4}{<(0,1)>}$ is 2 , so the qoutient must be isomorphic to $\mathbb{Z}_2$.

Exercise. 3

The order of $\mathbb{Z}_2 \times \mathbb{Z}_4$ is equal to 8 , and the order of (1,2) in $\mathbb{Z}_2 \times \mathbb{Z}_4$ is 2, hence the order of $\frac{\mathbb{Z}_2 \times \mathbb{Z}_4}{<(0,1)>}$ is 4 , so it must be isomorphic to either $\mathbb{Z}_4$, or $\mathbb{Z}_2 \times \mathbb{Z}_2$, but notice that all the elements in $\frac{\mathbb{Z}_2 \times \mathbb{Z}_4}{<(0,1)>}$ are of order 2, which implies that it is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.

Exercise. 7

Consider the map f defined from $\mathbb{Z} \times \mathbb{Z} \longrightarrow \mathbb{Z}$ by $f((a, b)) = 2a - b$
It is easy to see that f is a group homomorphism, which is surjective because $f(0, -n) = n$ for all $n \in \mathbb{Z}$
The kernel of $f = \{(a, b) \mid 2a = b\} = \{(a, 2a) \mid a \in \mathbb{R}\} = \{a(1, 2) \mid a \in \mathbb{R}\} =< (1, 2) >$.

Finally we deduce from the above that $\frac{\mathbb{Z} \times \mathbb{Z}}{<(1,2)>} \cong \mathbb{Z}$, since $\frac{\mathbb{Z} \times \mathbb{Z}}{Kerf} \cong Imf$

Exercise. 8

consider the map g defined from $\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z} \longrightarrow \mathbb{Z}\times\mathbb{Z}$ by $g((a,b,c)) = (a-b, a-c)$
It is easy to see that g is a group homomorphism, which is surjective because $g(0,-m,-n) = (m,n)$ for all $(m,n)\in\mathbb{Z}\times\mathbb{Z}$. The kernel of $g = \{(a,b,c) \mid a = b, a = c\} = \{(a,a,a) \mid a \in \mathbb{Z}\}$
$= \{a(1,1,1) \mid a \in \mathbb{R}\} =< (1,1,1) >$.

Finally we deduce from the above that $\frac{\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}}{<(1,1,1)>} \cong \mathbb{Z}\times\mathbb{Z}$ as in ex 7.

Exercise. 12

consider the map h defined from $\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z} \longrightarrow \mathbb{Z}_3\times\mathbb{Z}\times\mathbb{Z}$ by $h((a,b,c)) = (a+3\mathbb{Z}, a-b, a-c)$
It is easy to see that h is a group homomorphism, which is surjective because $h(m, m-n, m-p) = (m+3\mathbb{Z}, n, p)$. The kernel of $h = \{(a,b,c) \mid a = 0 \ \ mod(3), a = b, a = c\}$
$= \{(a,a,a) \mid a \in \mathbb{Z}, a \in 3\mathbb{Z}\} =< (3,3,3) >$.

Finally we deduce from the above that $\frac{\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}}{<(3,3,3)>} \cong \mathbb{Z}_3\times\mathbb{Z}\times\mathbb{Z}$ as in ex7.

Exercise. 35

N is a normal subgroup of G, we want to prove that $\phi(N)$ is normal in $\phi(G)$, so for any $y \in \phi(G)$ we need to prove $y\phi(N)y^{-1} \subset \phi(N)$.
So let $y \in \phi(G)$, and $n' \in \phi(N)$, we need to show that $yn'y^{-1} \in \phi(N)$, but we know that $y \in \phi(G)$ then there exist $x \in G$ such that $\phi(x) = y$ also since $n' \in \phi(N)$ there exists $n \in N$ such that $\phi(n) = n'$, hence $yn'y^{-1} = \phi(x)\phi(n)\phi(x)^{-1} = \phi(xnx^{-1}) \in \phi(N)$ since ($xnx^{-1} \in N$)

So we conclude that $\phi(N)$ is a normal subgroup of $\phi(G)$.

Exercise. 36

N' is normal, we need to prove that $\phi^{-1}(N')$ is normal, so we need to prove that $g\phi^{-1}(N')g^{-1} \subset \phi^{-1}(N')$ for all $g \in G$.
So let $g \in G$, and $n \in \phi^{-1}(N')$ then $\phi(g), \phi(n) \in G'$ then since N' is normal $\phi(g)\phi(n)\phi(g)^{-1} \in N'$, which implies that $\phi(gng^{-1}) \in N'$ which implies that $gng^{-1} \in \phi^{-1}(N')$.

So we conclude that $\phi^{-1}(N')$ is a normal subgroup.